NEIGHBORING FRACTIONS IN FAREY SUBSEQUENCES

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Abstract. We present explicit formulas for the computation of the neighbors of several elements of Farey subsequences.

1. Introduction

The *Farey sequence* $\mathcal{F}_m$ *of order* m is the ascending set of rational numbers $\frac{h}{k}$, written in reduced terms, such that $\frac{0}{1} \le \frac{h}{k} \le \frac{1}{1}$ and $1 \le k \le m$, see, e.g., [1, Chapter 27], [2], [3, §3], [4, Chapter 4], [5, Chapter III], [6, Chapter 6], [7, Chapter 6], [8, Sequences A006842 and A006843], [9, Chapter 5]. For example,

$$\mathcal{F}_5 = \left(\tfrac{0}{1} < \tfrac{1}{5} < \tfrac{1}{4} < \tfrac{1}{3} < \tfrac{2}{5} < \tfrac{1}{2} < \tfrac{3}{5} < \tfrac{2}{3} < \tfrac{3}{4} < \tfrac{4}{5} < \tfrac{1}{1}\right) .$$

Recall that the map

$$\mathcal{F}_m \to \mathcal{F}_m \ , \quad \tfrac{h}{k} \mapsto \tfrac{k-h}{k} \ , \tag{1}$$

is order-reversing and bijective.

Let $\overline{\mu}(\cdot)$ denote the Möbius function on positive integers. We use the notation $[s,t]$ to denote the interval $\{s, s+1, \ldots, t\}$ of positive integers; the greatest common divisor of s and t is denoted by $\gcd(s,t)$, and we write $s|t$ if t is divisible by s.

Notice that if y is a formal variable, then for a positive integer i and for a nonempty interval $[i'+1, i'']$ we have

$$\sum_{\substack{j \in [i'+1, i'']:\\ \gcd(i,j)=1}} \mathrm{y}^j = \sum_{\substack{d \in [1,i]:\\ d|i}} \overline{\mu}(d) \frac{\mathrm{y}^{d\left\lceil \frac{i'+1}{d} \right\rceil} - \mathrm{y}^{d \cdot (\left\lfloor \frac{i''}{d} \right\rfloor + 1)}}{1 - \mathrm{y}^d} \ .$$

Recall that $\bigl|\{j \in [i'+1, i''] : \gcd(i,j) = 1\}\bigr| = \sum_{d \in [1,i]:\ d|i} \overline{\mu}(d) \Bigl(\left\lfloor \frac{i''}{d} \right\rfloor - \left\lfloor \frac{i'}{d} \right\rfloor \Bigr)$, see, e.g., [10], [11]; see also, e.g., [12], [13] and references therein on relatively prime sets of integers and, in particular, on enumeration of l-subsets of $[i'+1, i'']$ that are relatively prime to i, for any l.

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If x and y are formal variables, then we have

$$\sum_{\substack{\frac{h}{k}\in\mathcal{F}_m:\\ \frac{0}{1}<\frac{h}{k}<\frac{1}{2}}} \mathrm{x}^h\mathrm{y}^k = \sum_{i\in[1,\lceil\frac{m}{2}\rceil-1]} \mathrm{x}^i \sum_{\substack{j\in[2i+1,m]:\\ \gcd(i,j)=1}} \mathrm{y}^j$$

$$= \sum_{i\in[1,\lceil\frac{m}{2}\rceil-1]} \mathrm{x}^i \sum_{\substack{d\in[1,i]:\\ d|i}} \overline{\mu}(d)\frac{\mathrm{y}^{d\lceil\frac{2i+1}{d}\rceil}-\mathrm{y}^{d\cdot(\lfloor\frac{m}{d}\rfloor+1)}}{1-\mathrm{y}^d} \ ;$$

thus,

$$\sum_{\substack{\frac{h}{k}\in\mathcal{F}_m:\\ \frac{0}{1}<\frac{h}{k}<\frac{1}{2}}} \mathrm{x}^h\mathrm{y}^k = \sum_{d\in[1,\lceil\frac{m}{2}\rceil-1]} \overline{\mu}(d)\frac{\mathrm{y}^d}{1-\mathrm{y}^d}\left(\frac{\mathrm{x}^d\mathrm{y}^{2d}-\mathrm{x}^{d\lceil\frac{m}{2d}\rceil}\mathrm{y}^{2d\lceil\frac{m}{2d}\rceil}}{1-\mathrm{x}^d\mathrm{y}^{2d}} - \frac{\mathrm{x}^d-\mathrm{x}^{d\lceil\frac{m}{2d}\rceil}}{1-\mathrm{x}^d}\mathrm{y}^{d\lfloor\frac{m}{d}\rfloor}\right) \ ;$$

for example,

$$\sum_{\substack{\frac{h}{k}\in\mathcal{F}_5:\\ \frac{0}{1}<\frac{h}{k}<\frac{1}{2}}} \mathrm{x}^h\mathrm{y}^k = \mathrm{x}\mathrm{y}^5+\mathrm{x}\mathrm{y}^4+\mathrm{x}\mathrm{y}^3+\mathrm{x}^2\mathrm{y}^5 = \frac{\mathrm{y}}{1-\mathrm{y}}\left(\frac{\mathrm{x}\mathrm{y}^2-\mathrm{x}^3\mathrm{y}^6}{1-\mathrm{x}\mathrm{y}^2} - \frac{\mathrm{x}-\mathrm{x}^3}{1-\mathrm{x}}\mathrm{y}^5\right) .$$

Similarly,

$$\sum_{\substack{\frac{h}{k}\in\mathcal{F}_m:\\ \frac{1}{2}<\frac{h}{k}<\frac{1}{1}}} \mathrm{x}^h\mathrm{y}^k = \sum_{d\in[1,\lceil\frac{m}{2}\rceil-1]} \overline{\mu}(d)\frac{\mathrm{x}^d\mathrm{y}^d}{1-\mathrm{x}^d\mathrm{y}^d}\left(\frac{\mathrm{x}^d\mathrm{y}^{2d}-\mathrm{x}^{d\lceil\frac{m}{2d}\rceil}\mathrm{y}^{2d\lceil\frac{m}{2d}\rceil}}{1-\mathrm{x}^d\mathrm{y}^{2d}} - \frac{\mathrm{x}^{d\cdot(\lfloor\frac{m}{d}\rfloor-\lceil\frac{m}{2d}\rceil+1)}-\mathrm{x}^{d\lfloor\frac{m}{d}\rfloor}}{1-\mathrm{x}^d}\mathrm{y}^{d\lfloor\frac{m}{d}\rfloor}\right) .$$

Standard tools for construction of the Farey sequences are numerical and matrix recurrences, as well as computational tree-like structures, see [4, Chapter 4].

The Farey sequence of order $2m$ contains the subsequence (see [10, 11, 14, 15])

$$\mathcal{F}\big(\mathbb{B}(2m),m\big) := \big(\tfrac{h}{k}\in\mathcal{F}_{2m} :\ k-m\leq h\leq m\big) \ , \tag{2}$$

which in a sense provides us with a dual description of the Farey sequence of order m, and vice versa; the notation $\mathbb{B}(2m)$ in definition (2) makes reference to the Boolean lattice of rank $2m$. Define the *left* and *right half-sequences* $\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)$ and $\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)$ of sequence (2) by

$$\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m),m\big) : = \big(\tfrac{h}{k}\in\mathcal{F}\big(\mathbb{B}(2m),m\big) :\ \tfrac{h}{k}\leq\tfrac{1}{2}\big)$$

and

$$\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m),m\big) := \big(\tfrac{h}{k}\in\mathcal{F}\big(\mathbb{B}(2m),m\big) :\ \tfrac{h}{k}\geq\tfrac{1}{2}\big)\ ,$$

respectively. The numerators of the fractions of $\mathcal{F}_m$ are the numerators of the fractions of $\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)$, while the denominators of the fractions of $\mathcal{F}_m$ are the numerators of the fractions of $\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)$. More precisely, the following can be said [11, 15]:

Lemma 1.1. *The maps*

$$\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)\to\mathcal{F}_m\ ,\qquad \tfrac{h}{k}\mapsto\tfrac{h}{k-h}\ ,\tag{3}$$

$$\mathcal{F}_m\to\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)\ ,\qquad \tfrac{h}{k}\mapsto\tfrac{h}{k+h}\ ,\tag{4}$$

$$\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)\to\mathcal{F}_m\ ,\qquad \tfrac{h}{k}\mapsto\tfrac{2h-k}{h}\ ,\tag{5}$$

and

$$\mathcal{F}_m\to\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)\ ,\qquad \tfrac{h}{k}\mapsto\tfrac{k}{2k-h}\ ,\tag{6}$$

are order-preserving and bijective.

The maps

$$\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)\to\mathcal{F}_m\ ,\qquad \tfrac{h}{k}\mapsto\tfrac{k-2h}{k-h}\ ,\tag{7}$$

$$\mathcal{F}_m\to\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)\ ,\qquad \tfrac{h}{k}\mapsto\tfrac{k-h}{2k-h}\ ,\tag{8}$$

$$\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)\to\mathcal{F}_m\ ,\qquad \tfrac{h}{k}\mapsto\tfrac{k-h}{h}\ ,\tag{9}$$

and

$$\mathcal{F}_m\to\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)\ ,\qquad \tfrac{h}{k}\mapsto\tfrac{k}{k+h}\ ,\tag{10}$$

are order-reversing and bijective.

For example, the sequence

$$\begin{aligned}\mathcal{F}\big(\mathbb{B}(10),5\big) = \big(\tfrac{0}{1}<\tfrac{1}{6}<\tfrac{1}{5}<\tfrac{1}{4}<\tfrac{2}{7}<\tfrac{1}{3}<\tfrac{3}{8}<\tfrac{2}{5}<\tfrac{3}{7}<\tfrac{4}{9}\\ <\tfrac{1}{2}<\tfrac{5}{9}<\tfrac{4}{7}<\tfrac{3}{5}<\tfrac{5}{8}<\tfrac{2}{3}<\tfrac{5}{7}<\tfrac{3}{4}<\tfrac{4}{5}<\tfrac{5}{6}<\tfrac{1}{1}\big)\end{aligned}$$

provides us with a dual description of the sequence $\mathcal{F}_5$, and vice versa.

Notice that the map

$$\mathcal{F}\big(\mathbb{B}(2m),m\big)\to\mathcal{F}\big(\mathbb{B}(2m),m\big)\ ,\quad \tfrac{h}{k}\mapsto\tfrac{k-h}{k}\ ,$$

is order-reversing and bijective.

Applications require a variety of pairs of adjacent fractions within the Farey (sub)sequences for the recurrent computation of other fractions to be performed efficiently; we are particularly interested in a family of pairs of adjacent fractions within $\mathcal{F}\big(\mathbb{B}(2m),m\big)$ which can be described more or less explicitly. We make use of machinery of elementary number theory; the approach is unified, and it consists in the transfer of the results of calculations

concerning $\mathcal{F}_m$ to $\mathcal{F}\bigl(\mathbb{B}(2m), m\bigr)$ by means of some monotone bijections collected in Lemma 1.1.

In Section 2 of the paper, we show how Farey (sub)sequences arise in analysis of collective decision-making procedures.

In Section 3, we first recall formulas for the computation of the neighbors of elements of $\mathcal{F}_m$ (Lemma 3.1); then, we present formulas that describe the neighbors of fractions of the form $\frac{1}{j}$, $\frac{j-1}{j}$ (Corollary 3.2), and of the form $\frac{2}{j}$, $\frac{j-2}{j}$ (Corollary 3.3).

In Section 4, we present formulas describing consecutive fractions in $\mathcal{F}\bigl(\mathbb{B}(2m), m\bigr)$ (Proposition 4.1), and we find the neighbors of fractions of the form $\frac{1}{j+1}$, $\frac{j-1}{2j-1}$, $\frac{j}{2j-1}$, $\frac{j}{j+1}$ (Corollary 4.2), and of the form $\frac{2}{j+2}$, $\frac{j-2}{2(j-1)}$, $\frac{j}{2(j-1)}$, $\frac{j}{j+2}$ (Corollary 4.3).

In Section 5, we slightly simplify our calculations, made in Section 4, by describing three subsequences of fractions that are successive in $\mathcal{F}\bigl(\mathbb{B}(2m), m\bigr)$.

In the paper, we consider Farey (sub)sequences $\mathcal{F}_m$ and $\mathcal{F}\bigl(\mathbb{B}(2m), m\bigr)$ such that $m > 1$.

2. Farey subsequences and collective decision making

The Farey sequence $\mathcal{F}_m$ has a wide area of applications in mathematics, computer science and physics [2]. The Farey subsequence $\mathcal{F}\bigl(\mathbb{B}(2m), m\bigr)$ which is closely related to $\mathcal{F}_m$, as described in Lemma 1.1, may be useful for analysis of procedures of collective decision making. One such procedure consists in pattern recognition with the help of committee decision rules, see [16] and references therein.

A finite collection $\boldsymbol{\mathcal{H}}$ of pairwise distinct hyperplanes in the *feature space* $\mathbb{R}^n$ is a *training set* if it is partitioned into two nonempty subsets $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$. A codimension one subspace $\boldsymbol{H} := \{\mathrm{x} \in \mathbb{R}^n : \ \langle \boldsymbol{p}, \mathrm{x}\rangle := \sum_{j=1}^{n} p_j \mathrm{x}_j = 0\}$ from the *hyperplane arrangement* $\boldsymbol{\mathcal{H}}$ is defined by its normal vector $\boldsymbol{p} \in \mathbb{R}^n$, and this hyperplane is oriented: a vector $\boldsymbol{v}$ lies on the *positive side* of $\boldsymbol{H}$ if $\langle \boldsymbol{h}, \boldsymbol{v}\rangle > 0$, where, by convention, $\boldsymbol{h} := -\boldsymbol{p}$ if $\boldsymbol{H} \in \boldsymbol{\mathcal{A}}$, and $\boldsymbol{h} := \boldsymbol{p}$ if $\boldsymbol{H} \in \boldsymbol{\mathcal{B}}$. In a similar manner, a *region* $\boldsymbol{T}$ of the hyperplane arrangement $\boldsymbol{\mathcal{H}}$, that is, a connected component of the *complement* $\boldsymbol{\mathcal{T}} := \mathbb{R}^n - \boldsymbol{\mathcal{H}}$, lies on the positive side of the hyperplane $\boldsymbol{H}$ if $\langle \boldsymbol{h}, \boldsymbol{v}\rangle > 0$ for some vector $\boldsymbol{v} \in \boldsymbol{T}$. Let $\boldsymbol{\mathcal{T}}_{\boldsymbol{H}}^{+}$ denote the set of all regions lying on the positive side of $\boldsymbol{H}$.

The oriented hyperplanes from the arrangement $\boldsymbol{\mathcal{H}}$ are called the *training patterns*. The *training samples* $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ provide a partial description of two disjont *classes* $\mathbf{A}$ and $\mathbf{B}$, respectively: *a priori*, we have $\mathbf{A} \supseteq \boldsymbol{\mathcal{A}}$ and $\mathbf{B} \supseteq \boldsymbol{\mathcal{B}}$.

A subset $\boldsymbol{\mathcal{K}}^* \subset \boldsymbol{\mathcal{T}}$ is a *committee of regions* for the arrangement $\boldsymbol{\mathcal{H}}$ if $|\boldsymbol{\mathcal{K}}^* \cap \boldsymbol{\mathcal{T}}_{\boldsymbol{H}}^{+}| > \frac{1}{2}|\boldsymbol{\mathcal{K}}^*|$, for each hyperplane $\boldsymbol{H} \in \boldsymbol{\mathcal{H}}$.

Consider a new pattern $\boldsymbol{G} := \{\mathrm{x} \in \mathbb{R}^n : \ \langle \boldsymbol{g}, \mathrm{x}\rangle = 0\} \not\in \boldsymbol{\mathcal{H}}$, defined by its normal vector $\boldsymbol{g} \in \mathbb{R}^n$. If a system of distinct representatives $\boldsymbol{W} := \{\boldsymbol{w} \in \boldsymbol{K} : \ \boldsymbol{K} \in \boldsymbol{\mathcal{K}}^*\}$, of cardinality $|\boldsymbol{\mathcal{K}}^*|$, for the committee of regions $\boldsymbol{\mathcal{K}}^*$ is fixed, then the corresponding *committee decision rule* recognizes the pattern $\boldsymbol{G}$ as

an element of the class $\mathbf{A}$ if $|\{\boldsymbol{w}\in\boldsymbol{W}:\ \langle\boldsymbol{g},\boldsymbol{w}\rangle>0\}|<\frac{1}{2}|\boldsymbol{W}|$; the pattern $\boldsymbol{G}$ is recognized as an element of the class $\mathbf{B}$ if $|\{\boldsymbol{w}\in\boldsymbol{W}:\ \langle\boldsymbol{g},\boldsymbol{w}\rangle>0\}|>\frac{1}{2}|\boldsymbol{W}|$.

For any hyperplane $\boldsymbol{H}$ from the arrangement $\boldsymbol{\mathcal{H}}$, the ascending collection of irreducible fractions

$$\left(\frac{|\boldsymbol{\mathcal{R}}\cap\boldsymbol{\mathcal{T}}^+_{\boldsymbol{H}}|}{\gcd(|\boldsymbol{\mathcal{R}}\cap\boldsymbol{\mathcal{T}}^+_{\boldsymbol{H}}|,|\boldsymbol{\mathcal{R}}|)}\Big/\frac{|\boldsymbol{\mathcal{R}}|}{\gcd(|\boldsymbol{\mathcal{R}}\cap\boldsymbol{\mathcal{T}}^+_{\boldsymbol{H}}|,|\boldsymbol{\mathcal{R}}|)}:\ \boldsymbol{\mathcal{R}}\subseteq\boldsymbol{\mathcal{T}},\ |\boldsymbol{\mathcal{R}}|>0\right)$$

is the Farey subsequence $\mathcal{F}\big(\mathbb{B}(|\boldsymbol{\mathcal{T}}|),\frac{|\boldsymbol{\mathcal{T}}|}{2}\big)$, a dual of the standard Farey sequence $\mathcal{F}_{|\boldsymbol{\mathcal{T}}|/2}$. A neighborhood of the critical value $\frac{1}{2}$ in $\mathcal{F}\big(\mathbb{B}(|\boldsymbol{\mathcal{T}}|),\frac{|\boldsymbol{\mathcal{T}}|}{2}\big)$ has the simple structure explained in Remark 5.1(ii)(iii) of Section 5. From the number-theoretic point of view, a subset of regions $\boldsymbol{\mathcal{K}}^*\subset\boldsymbol{\mathcal{T}}$ is a committee for the arrangement $\boldsymbol{\mathcal{H}}$ if and only if for each hyperplane $\boldsymbol{H}\in\boldsymbol{\mathcal{H}}$ it holds

$$\frac{|\boldsymbol{\mathcal{K}}^*\cap\boldsymbol{\mathcal{T}}^+_{\boldsymbol{H}}|}{\gcd(|\boldsymbol{\mathcal{K}}^*\cap\boldsymbol{\mathcal{T}}^+_{\boldsymbol{H}}|,|\boldsymbol{\mathcal{K}}^*|)}\Big/\frac{|\boldsymbol{\mathcal{K}}^*|}{\gcd(|\boldsymbol{\mathcal{K}}^*\cap\boldsymbol{\mathcal{T}}^+_{\boldsymbol{H}}|,|\boldsymbol{\mathcal{K}}^*|)}\in\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(|\boldsymbol{\mathcal{T}}|),\tfrac{|\boldsymbol{\mathcal{T}}|}{2}\big)-\{\tfrac{1}{2}\}\ ;$$

thus, the study of the structure of the family of all committees of regions for the hyperplane arrangement $\boldsymbol{\mathcal{H}}$ might involve the sequence $\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(|\boldsymbol{\mathcal{T}}|),\frac{|\boldsymbol{\mathcal{T}}|}{2}\big)$. Such an attempt is made in [15] within the bounds of *oriented matroid* theory.

3. Neighbors in $\mathcal{F}_m$

We begin this section by recalling several observations made in elementary number theory, cf. [1, Chapter 27], [5, Chapter III]:

• Consider a fraction $\frac{h}{k}\in\mathcal{F}_m-\{\frac{0}{1}\}$. To find the fraction that precedes $\frac{h}{k}$, consider any integer x_0 such that

$$kx_0\equiv-1\pmod{h}\ ,$$

and $y_0:=\frac{kx_0+1}{h}$. For any integer t, the pair (x_0+ht,y_0+kt) is a solution to the Diophantine equation $h\mathrm{y}-k\mathrm{x}=1$. The integer $t^*:=\left\lfloor\frac{hm-kx_0-1}{hk}\right\rfloor$ is the maximum solution to the inequality system

$$0\leq x_0+h\mathrm{t}\leq m\ ,\quad 1\leq y_0+k\mathrm{t}\leq m\ ; \tag{11}$$

we have $\frac{x_0+ht^*}{y_0+kt^*}\in\mathcal{F}_m$. Notice that for any integer solutions t' and t'' to system (11), such that $t'\leq t''$, it holds $\frac{0}{1}\leq\frac{x_0+ht'}{y_0+kt'}\leq\frac{x_0+ht''}{y_0+kt''}<\frac{h}{k}$. Since there is no fraction $\frac{i}{j}\in\mathcal{F}_m$ such that $\frac{x_0+ht^*}{y_0+kt^*}<\frac{i}{j}<\frac{h}{k}$, the fraction

$$\left(x_0+h\left\lfloor\frac{hm-kx_0-1}{hk}\right\rfloor\right)\Big/\left(\frac{kx_0+1}{h}+k\left\lfloor\frac{hm-kx_0-1}{hk}\right\rfloor\right) \tag{12}$$

precedes $\frac{h}{k}$ in $\mathcal{F}_m$. The multiplier $\left\lfloor\frac{hm-kx_0-1}{hk}\right\rfloor$ in formula (12) turns into zero whenever $\lceil\frac{hm}{k}\rceil-h\leq x_0\leq\lceil\frac{hm}{k}\rceil-1$.

Now, if y_0 is an integer such that

$$hy_0\equiv1\pmod{k}\ ,$$

then the fraction

$$\left(\tfrac{hy_0-1}{k} + h\left\lfloor\tfrac{m-y_0}{k}\right\rfloor\right)\Big/\left(y_0 + k\left\lfloor\tfrac{m-y_0}{k}\right\rfloor\right) \ , \tag{13}$$

which coincides with the fraction described by formula (12), precedes $\frac{h}{k}$ in $\mathcal{F}_m$.

• Consider a fraction $\frac{h}{k} \in \mathcal{F}_m - \{\frac{1}{1}\}$.

If x_0 and y_0 are integers such that

$$\begin{aligned} kx_0 &\equiv 1 \pmod{h} \ , \\ hy_0 &\equiv -1 \pmod{k} \ , \end{aligned}$$

then the fraction

$$\begin{aligned} &\left(x_0 + h\left\lfloor\tfrac{hm-kx_0+1}{hk}\right\rfloor\right)\Big/\left(\tfrac{kx_0-1}{h} + k\left\lfloor\tfrac{hm-kx_0+1}{hk}\right\rfloor\right) && (14)\\ &= \left(\tfrac{hy_0+1}{k} + h\left\lfloor\tfrac{m-y_0}{k}\right\rfloor\right)\Big/\left(y_0 + k\left\lfloor\tfrac{m-y_0}{k}\right\rfloor\right) && (15) \end{aligned}$$

succeeds $\frac{h}{k}$ in $\mathcal{F}_m$.

Recall that $\frac{m-1}{m}$ precedes $\frac{1}{1}$, and $\frac{1}{m}$ succeeds $\frac{0}{1}$ in $\mathcal{F}_m$. The following proposition lists simplified versions of expressions (12), (13), (14) and (15) for the neighbors of fractions in $\mathcal{F}_m$:

Lemma 3.1. (i) *Consider a fraction $\frac{h}{k} \in \mathcal{F}_m - \{\frac{0}{1}\}$. Let a and b be the integers such that*

$$\begin{aligned} ka &\equiv -1 \pmod{h} \ , && \left\lceil\tfrac{hm}{k}\right\rceil - h \le a \le \left\lceil\tfrac{hm}{k}\right\rceil - 1 \ , && (16)\\ hb &\equiv 1 \pmod{k} \ , && m-k+1 \le b \le m \ . \end{aligned}$$

The fraction

$$a\Big/\tfrac{ka+1}{h} \quad = \quad \tfrac{hb-1}{k}\Big/b$$

precedes $\frac{h}{k}$ in $\mathcal{F}_m$.

(ii) *Consider a fraction $\frac{h}{k} \in \mathcal{F}_m - \{\frac{1}{1}\}$. Let a and b be the integers such that*

$$\begin{aligned} ka &\equiv 1 \pmod{h} \ , && \left\lceil\tfrac{hm+2}{k}\right\rceil - h \le a \le \left\lceil\tfrac{hm+2}{k}\right\rceil - 1 \ , && (17)\\ hb &\equiv -1 \pmod{k} \ , && m-k+1 \le b \le m \ . \end{aligned}$$

The fraction

$$a\Big/\tfrac{ka-1}{h} \quad = \quad \tfrac{hb+1}{k}\Big/b$$

succeeds $\frac{h}{k}$ in $\mathcal{F}_m$.

Inspired by [1, Theorem 253], our approach to the search for the neighbors of a general fraction $\frac{h}{k}$ in $\mathcal{F}_m$ is more flexible: while the left boundary of the search interval in [1, Theorem 253] is determined by the denominator k, the left boundaries of our additional search intervals included in constraints (16) and (17) depend on the numerator h; as a consequence, in

many cases these additional search intervals are much more shorter. Moreover, if we consider in Lemma 3.1 fractions of the form $\frac{1}{j}$ then the search intervals for integers a in constraints (16) and (17) turn into singletons:

Corollary 3.2. (i) *Consider a fraction $\frac{1}{j} \in \mathcal{F}_m$. The fraction*

$$\frac{\left\lceil \frac{m}{j} \right\rceil - 1}{j \cdot \left(\left\lceil \frac{m}{j} \right\rceil - 1 \right) + 1} \tag{18}$$

precedes $\frac{1}{j}$ in $\mathcal{F}_m$. If $j > 1$, then the fraction

$$\frac{\left\lceil \frac{m+2}{j} \right\rceil - 1}{j \cdot \left(\left\lceil \frac{m+2}{j} \right\rceil - 1 \right) - 1}$$

succeeds $\frac{1}{j}$ in $\mathcal{F}_m$.

(ii) *Consider a fraction $\frac{j-1}{j} \in \mathcal{F}_m$. If $j > 1$, then the fraction*

$$\frac{(j-1)\left(\left\lceil \frac{m+2}{j} \right\rceil - 1 \right) - 1}{j \cdot \left(\left\lceil \frac{m+2}{j} \right\rceil - 1 \right) - 1}$$

precedes $\frac{j-1}{j}$ in $\mathcal{F}_m$. The fraction

$$\frac{(j-1)\left(\left\lceil \frac{m}{j} \right\rceil - 1 \right) + 1}{j \cdot \left(\left\lceil \frac{m}{j} \right\rceil - 1 \right) + 1}$$

succeeds $\frac{j-1}{j}$ in $\mathcal{F}_m$.

To prove assertion (ii) of Corollary 3.2, notice that $\frac{1}{j}$ is the image of $\frac{j-1}{j}$ under bijection (1), and find the images of the neighbors of $\frac{1}{j}$ under map (1).

The following statement summarizes the results of similar calculations that can be performed, with the help of Lemma 3.1, for fractions of the form $\frac{2}{j}$, $\frac{j-2}{j} \in \mathcal{F}_m$:

Corollary 3.3. (i) *If $\frac{2}{j} \in \mathcal{F}_m$, for some j, then the fraction*

$$\begin{cases} \left(\left\lceil \frac{2m}{j} \right\rceil - 1 \right) \Big/ \frac{j \cdot \left(\left\lceil \frac{2m}{j} \right\rceil - 1 \right) + 1}{2} , & \text{if } \left\lceil \frac{2m}{j} \right\rceil \equiv 0 \pmod{2} \ ; \\ \left(\left\lceil \frac{2m}{j} \right\rceil - 2 \right) \Big/ \frac{j \cdot \left(\left\lceil \frac{2m}{j} \right\rceil - 2 \right) + 1}{2} , & \text{if } \left\lceil \frac{2m}{j} \right\rceil \equiv 1 \pmod{2} \ ; \end{cases}$$

precedes $\frac{2}{j}$ *in* $\mathcal{F}_m$*; the fraction*

$$\begin{cases} \left(\left\lceil\frac{2(m+1)}{j}\right\rceil - 1\right)\Big/\frac{j\cdot\left(\left\lceil\frac{2(m+1)}{j}\right\rceil-1\right)-1}{2} \;, & \text{if } \left\lceil\frac{2(m+1)}{j}\right\rceil \equiv 0 \pmod 2 \;; \\ \left(\left\lceil\frac{2(m+1)}{j}\right\rceil - 2\right)\Big/\frac{j\cdot\left(\left\lceil\frac{2(m+1)}{j}\right\rceil-2\right)-1}{2} \;, & \text{if } \left\lceil\frac{2(m+1)}{j}\right\rceil \equiv 1 \pmod 2 \;; \end{cases}$$

succeeds $\frac{2}{j}$ *in* $\mathcal{F}_m$*.*

(ii) *If* $\frac{j-2}{j} \in \mathcal{F}_m$*, for some* j*, then the fraction*

$$\begin{cases} \frac{(j-2)\left(\left\lceil\frac{2(m+1)}{j}\right\rceil-1\right)-1}{2}\Big/\frac{j\cdot\left(\left\lceil\frac{2(m+1)}{j}\right\rceil-1\right)-1}{2} \;, & \text{if } \left\lceil\frac{2(m+1)}{j}\right\rceil \equiv 0 \pmod 2 \;; \\ \frac{(j-2)\left(\left\lceil\frac{2(m+1)}{j}\right\rceil-2\right)-1}{2}\Big/\frac{j\cdot\left(\left\lceil\frac{2(m+1)}{j}\right\rceil-2\right)-1}{2} \;, & \text{if } \left\lceil\frac{2(m+1)}{j}\right\rceil \equiv 1 \pmod 2 \;; \end{cases}$$

precedes $\frac{j-2}{j}$ *in* $\mathcal{F}_m$*; the fraction*

$$\begin{cases} \frac{(j-2)\left(\left\lceil\frac{2m}{j}\right\rceil-1\right)+1}{2}\Big/\frac{j\cdot\left(\left\lceil\frac{2m}{j}\right\rceil-1\right)+1}{2} \;, & \text{if } \left\lceil\frac{2m}{j}\right\rceil \equiv 0 \pmod 2 \;; \\ \frac{(j-2)\left(\left\lceil\frac{2m}{j}\right\rceil-2\right)+1}{2}\Big/\frac{j\cdot\left(\left\lceil\frac{2m}{j}\right\rceil-2\right)+1}{2} \;, & \text{if } \left\lceil\frac{2m}{j}\right\rceil \equiv 1 \pmod 2 \;; \end{cases}$$

succeeds $\frac{j-2}{j}$ *in* $\mathcal{F}_m$*.*

4. Neighbors in $\mathcal{F}\big(\mathbb{B}(2m), m\big)$

We now extend the results, obtained in the previous section, to the Farey subsequence $\mathcal{F}\big(\mathbb{B}(2m), m\big)$. We begin by searching for the neighbors of an arbitrary fraction in $\mathcal{F}\big(\mathbb{B}(2m), m\big)$:

Proposition 4.1. (i) *Consider a fraction* $\frac{h}{k} \in \mathcal{F}^{\leq \frac{1}{2}}\big(\mathbb{B}(2m), m\big)$.

(a) *If* $\frac{h}{k} \neq \frac{0}{1}$*, let* a *and* b *be the integers such that*

$$(k-h)a \equiv -1 \pmod h \;, \qquad \left\lceil\frac{hm}{k-h}\right\rceil - h \leq a \leq \left\lceil\frac{hm}{k-h}\right\rceil - 1 \;,$$
$$hb \equiv 1 \pmod{(k-h)} \;, \qquad m-k+h+1 \leq b \leq m \;.$$

The fraction

$$a\Big/\frac{ka+1}{h} \quad = \quad \frac{hb-1}{k-h}\Big/\frac{kb-1}{k-h}$$

precedes $\frac{h}{k}$ *in* $\mathcal{F}\big(\mathbb{B}(2m), m\big)$*.*

(b) *If* $\frac{h}{k} \neq \frac{1}{2}$*, let* a *and* b *be the integers such that*

$$(k-h)a \equiv 1 \pmod h \;, \qquad \left\lceil\frac{hm+2}{k-h}\right\rceil - h \leq a \leq \left\lceil\frac{hm+2}{k-h}\right\rceil - 1 \;,$$
$$hb \equiv -1 \pmod{(k-h)} \;, \qquad m-k+h+1 \leq b \leq m \;.$$

The fraction

$$a\Big/\frac{ka-1}{h} \quad = \quad \frac{hb+1}{k-h}\Big/\frac{kb+1}{k-h}$$

succeeds $\frac{h}{k}$ *in* $\mathcal{F}\big(\mathbb{B}(2m),m\big)$.

(ii) *Consider a fraction* $\frac{h}{k}\in\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(2m),m\big)$.

(a) *If* $\frac{h}{k}\neq\frac{1}{2}$, *let* a *and* b *be the integers such that*

$$ka\equiv -1 \pmod{h}\ , \qquad m-h+1\leq a\leq m\ ,$$
$$hb\equiv 1 \pmod{(k-h)}\ , \qquad \left\lceil\tfrac{(k-h)m+2}{h}\right\rceil-k+h\leq b\leq\left\lceil\tfrac{(k-h)m+2}{h}\right\rceil-1\ .$$

The fraction

$$a\Big/\tfrac{ka+1}{h} \quad = \quad \tfrac{hb-1}{k-h}\Big/\tfrac{kb-1}{k-h}$$

precedes $\frac{h}{k}$ *in* $\mathcal{F}\big(\mathbb{B}(2m),m\big)$.

(b) *If* $\frac{h}{k}\neq\frac{1}{1}$, *let* a *and* b *be the integers such that*

$$ka\equiv 1 \pmod{h}\ , \qquad m-h+1\leq a\leq m\ ,$$
$$hb\equiv -1 \pmod{(k-h)}\ , \qquad \left\lceil\tfrac{(k-h)m}{h}\right\rceil-k+h\leq b\leq\left\lceil\tfrac{(k-h)m}{h}\right\rceil-1\ .$$

The fraction

$$a\Big/\tfrac{ka-1}{h} \quad = \quad \tfrac{hb+1}{k-h}\Big/\tfrac{kb+1}{k-h}$$

succeeds $\frac{h}{k}$ *in* $\mathcal{F}\big(\mathbb{B}(2m),m\big)$.

Proof. To prove assertion (i)(a), notice that $\frac{h}{k-h}\in\mathcal{F}_m$ is the image of $\frac{h}{k}$ under bijection (3), use Lemma 3.1(i) to find the predecessor of $\frac{h}{k-h}$ in $\mathcal{F}_m$, and send it to $\mathcal{F}\big(\mathbb{B}(2m),m\big)$ by means of bijection (4).

Assertion (i)(b) is proved in a similar way, by the application of Lemma 3.1(ii), with the help of bijections (3) and (4).

One can prove Proposition 4.1(ii) by the application of Lemma 3.1, with the help of bijections (9) and (10). $\square$

Corollary 4.2. (i) *If* $\frac{1}{j+1}\in\mathcal{F}\big(\mathbb{B}(2m),m\big)$ *and* $\frac{1}{j+1}<\frac{1}{2}$, *for some* j, *then the fraction*

$$\frac{\left\lceil\frac{m}{j}\right\rceil-1}{(j+1)\left(\left\lceil\frac{m}{j}\right\rceil-1\right)+1}$$

precedes $\frac{1}{j+1}$ *in* $\mathcal{F}\big(\mathbb{B}(2m),m\big)$; *the fraction*

$$\frac{\left\lceil\frac{m+2}{j}\right\rceil-1}{(j+1)\left(\left\lceil\frac{m+2}{j}\right\rceil-1\right)-1}$$

succeeds $\frac{1}{j+1}$ *in* $\mathcal{F}\big(\mathbb{B}(2m),m\big)$.

(ii) *If* $\frac{j-1}{2j-1} \in \mathcal{F}\big(\mathbb{B}(2m), m\big)$ *and* $\frac{j-1}{2j-1} < \frac{1}{2}$, *for some* j, *then the fraction*

$$\frac{(j-1)\left(\left\lceil\frac{m+2}{j}\right\rceil - 1\right) - 1}{(2j-1)\left(\left\lceil\frac{m+2}{j}\right\rceil - 1\right) - 2}$$

precedes $\frac{j-1}{2j-1}$ *in* $\mathcal{F}\big(\mathbb{B}(2m), m\big)$; *the fraction*

$$\frac{(j-1)\left(\left\lceil\frac{m}{j}\right\rceil - 1\right) + 1}{(2j-1)\left(\left\lceil\frac{m}{j}\right\rceil - 1\right) + 2}$$

succeeds $\frac{j-1}{2j-1}$ *in* $\mathcal{F}\big(\mathbb{B}(2m), m\big)$.

(iii) *If* $\frac{j}{2j-1} \in \mathcal{F}\big(\mathbb{B}(2m), m\big)$ *and* $\frac{1}{2} < \frac{j}{2j-1} < \frac{1}{1}$, *for some* j, *then the fraction*

$$\frac{j \cdot \left(\left\lceil\frac{m}{j}\right\rceil - 1\right) + 1}{(2j-1)\left(\left\lceil\frac{m}{j}\right\rceil - 1\right) + 2}$$

precedes $\frac{j}{2j-1}$ *in* $\mathcal{F}\big(\mathbb{B}(2m), m\big)$; *the fraction*

$$\frac{j \cdot \left(\left\lceil\frac{m+2}{j}\right\rceil - 1\right) - 1}{(2j-1)\left(\left\lceil\frac{m+2}{j}\right\rceil - 1\right) - 2}$$

succeeds $\frac{j}{2j-1}$ *in* $\mathcal{F}\big(\mathbb{B}(2m), m\big)$.

(iv) *If* $\frac{j}{j+1} \in \mathcal{F}\big(\mathbb{B}(2m), m\big)$ *and* $\frac{1}{2} < \frac{j}{j+1}$, *for some* j, *then the fraction*

$$\frac{j \cdot \left(\left\lceil\frac{m+2}{j}\right\rceil - 1\right) - 1}{(j+1)\left(\left\lceil\frac{m+2}{j}\right\rceil - 1\right) - 1}$$

precedes $\frac{j}{j+1}$ *in* $\mathcal{F}\big(\mathbb{B}(2m), m\big)$; *the fraction*

$$\frac{j \cdot \left(\left\lceil\frac{m}{j}\right\rceil - 1\right) + 1}{(j+1)\left(\left\lceil\frac{m}{j}\right\rceil - 1\right) + 1}$$

succeeds $\frac{j}{j+1}$ *in* $\mathcal{F}\big(\mathbb{B}(2m), m\big)$.

Proof. The fraction $\frac{1}{j} \in \mathcal{F}_m$ is the image of the fractions $\frac{1}{j+1}$, $\frac{j-1}{2j-1}$, $\frac{j}{2j-1}$ and $\frac{j}{j+1}$ under bijections (3), (7), (5) and (9), respectively.

To prove assertion (i), find the predecessor of $\frac{1}{j}$ in $\mathcal{F}_m$ by the application of Corollary 3.2(i), and reflect it to $\mathcal{F}\big(\mathbb{B}(2m), m\big)$ by means of bijection (4).

One can prove the remaining assertions in a similar way, with the help of Corollary 3.2 and of the monotone bijections mentioned in Lemma 1.1. □

Recall also that the fraction $\frac{m}{m+1}$ precedes $\frac{1}{1}$, and $\frac{1}{m+1}$ succeeds $\frac{0}{1}$ in $\mathcal{F}\bigl(\mathbb{B}(2m),m\bigr)$; the three fractions $\frac{m-1}{2m-1} < \frac{1}{2} < \frac{m}{2m-1}$ are successive in $\mathcal{F}\bigl(\mathbb{B}(2m),m\bigr)$, see [15].

Corollary 4.3. (i) *If $\frac{2}{j+2} \in \mathcal{F}\bigl(\mathbb{B}(2m),m\bigr)$ and $\frac{2}{j+2} < \frac{1}{2}$, for some j, then the fraction*

$$\begin{cases} \left(\left\lceil \frac{2m}{j} \right\rceil - 1\right) \Big/ \frac{(j+2)\left(\left\lceil \frac{2m}{j} \right\rceil - 1\right)+1}{2} \;, & \text{if } \left\lceil \frac{2m}{j} \right\rceil \equiv 0 \pmod 2 \;; \\ \left(\left\lceil \frac{2m}{j} \right\rceil - 2\right) \Big/ \frac{(j+2)\left(\left\lceil \frac{2m}{j} \right\rceil - 2\right)+1}{2} \;, & \text{if } \left\lceil \frac{2m}{j} \right\rceil \equiv 1 \pmod 2 \;; \end{cases}$$

precedes $\frac{2}{j+2}$ in $\mathcal{F}\bigl(\mathbb{B}(2m),m\bigr)$; the fraction

$$\begin{cases} \left(\left\lceil \frac{2(m+1)}{j} \right\rceil - 1\right) \Big/ \frac{(j+2)\left(\left\lceil \frac{2(m+1)}{j} \right\rceil - 1\right)-1}{2} \;, & \text{if } \left\lceil \frac{2(m+1)}{j} \right\rceil \equiv 0 \pmod 2 \;; \\ \left(\left\lceil \frac{2(m+1)}{j} \right\rceil - 2\right) \Big/ \frac{(j+2)\left(\left\lceil \frac{2(m+1)}{j} \right\rceil - 2\right)-1}{2} \;, & \text{if } \left\lceil \frac{2(m+1)}{j} \right\rceil \equiv 1 \pmod 2 \;; \end{cases}$$

succeeds $\frac{2}{j+2}$ in $\mathcal{F}\bigl(\mathbb{B}(2m),m\bigr)$.

(ii) *If $\frac{j-2}{2(j-1)} \in \mathcal{F}\bigl(\mathbb{B}(2m),m\bigr)$ and $\frac{j-2}{2(j-1)} < \frac{1}{2}$, for some j, then the fraction*

$$\begin{cases} \frac{(j-2)\left(\left\lceil \frac{2(m+1)}{j} \right\rceil - 1\right)-1}{2} \Big/ \left((j-1)\left(\left\lceil \frac{2(m+1)}{j} \right\rceil - 1\right) - 1\right) \;, & \text{if } \left\lceil \frac{2(m+1)}{j} \right\rceil \equiv 0 \pmod 2 \;; \\ \frac{(j-2)\left(\left\lceil \frac{2(m+1)}{j} \right\rceil - 2\right)-1}{2} \Big/ \left((j-1)\left(\left\lceil \frac{2(m+1)}{j} \right\rceil - 2\right) - 1\right) \;, & \text{if } \left\lceil \frac{2(m+1)}{j} \right\rceil \equiv 1 \pmod 2 \;; \end{cases}$$

precedes $\frac{j-2}{2(j-1)}$ in $\mathcal{F}\bigl(\mathbb{B}(2m),m\bigr)$; the fraction

$$\begin{cases} \frac{(j-2)\left(\left\lceil \frac{2m}{j} \right\rceil - 1\right)+1}{2} \Big/ \left((j-1)\left(\left\lceil \frac{2m}{j} \right\rceil - 1\right) + 1\right) \;, & \text{if } \left\lceil \frac{2m}{j} \right\rceil \equiv 0 \pmod 2 \;; \\ \frac{(j-2)\left(\left\lceil \frac{2m}{j} \right\rceil - 2\right)+1}{2} \Big/ \left((j-1)\left(\left\lceil \frac{2m}{j} \right\rceil - 2\right) + 1\right) \;, & \text{if } \left\lceil \frac{2m}{j} \right\rceil \equiv 1 \pmod 2 \;; \end{cases}$$

succeeds $\frac{j-2}{2(j-1)}$ in $\mathcal{F}\bigl(\mathbb{B}(2m),m\bigr)$.

(iii) *If $\frac{j}{2(j-1)} \in \mathcal{F}\bigl(\mathbb{B}(2m),m\bigr)$ and $\frac{1}{2} < \frac{j}{2(j-1)}$, for some j, then the fraction*

$$\begin{cases} \frac{j\cdot\left(\left\lceil \frac{2m}{j} \right\rceil - 1\right)+1}{2} \Big/ \left((j-1)\left(\left\lceil \frac{2m}{j} \right\rceil - 1\right) + 1\right) \;, & \text{if } \left\lceil \frac{2m}{j} \right\rceil \equiv 0 \pmod 2 \;; \\ \frac{j\cdot\left(\left\lceil \frac{2m}{j} \right\rceil - 2\right)+1}{2} \Big/ \left((j-1)\left(\left\lceil \frac{2m}{j} \right\rceil - 2\right) + 1\right) \;, & \text{if } \left\lceil \frac{2m}{j} \right\rceil \equiv 1 \pmod 2 \;; \end{cases}$$

precedes $\frac{j}{2(j-1)}$ *in* $\mathcal{F}\bigl(\mathbb{B}(2m),m\bigr)$*; the fraction*

$$\begin{cases}\frac{j\cdot\left(\left\lceil\frac{2(m+1)}{j}\right\rceil-1\right)-1}{2}\Big/\left((j-1)\left(\left\lceil\frac{2(m+1)}{j}\right\rceil-1\right)-1\right)\ , & \textit{if } \left\lceil\frac{2(m+1)}{j}\right\rceil\equiv 0 \pmod 2\ ;\\ \frac{j\cdot\left(\left\lceil\frac{2(m+1)}{j}\right\rceil-2\right)-1}{2}\Big/\left((j-1)\left(\left\lceil\frac{2(m+1)}{j}\right\rceil-2\right)-1\right)\ , & \textit{if } \left\lceil\frac{2(m+1)}{j}\right\rceil\equiv 1 \pmod 2\ ;\end{cases}$$

succeeds $\frac{j}{2(j-1)}$ *in* $\mathcal{F}\bigl(\mathbb{B}(2m),m\bigr)$*.*

(iv) *If* $\frac{j}{j+2}\in\mathcal{F}\bigl(\mathbb{B}(2m),m\bigr)$ *and* $\frac{1}{2}<\frac{j}{j+2}$*, for some* j*, then the fraction*

$$\begin{cases}\frac{j\cdot\left(\left\lceil\frac{2(m+1)}{j}\right\rceil-1\right)-1}{2}\Big/\frac{(j+2)\left(\left\lceil\frac{2(m+1)}{j}\right\rceil-1\right)-1}{2}\ , & \textit{if } \left\lceil\frac{2(m+1)}{j}\right\rceil\equiv 0 \pmod 2\ ;\\ \frac{j\cdot\left(\left\lceil\frac{2(m+1)}{j}\right\rceil-2\right)-1}{2}\Big/\frac{(j+2)\left(\left\lceil\frac{2(m+1)}{j}\right\rceil-2\right)-1}{2}\ , & \textit{if } \left\lceil\frac{2(m+1)}{j}\right\rceil\equiv 1 \pmod 2\ ;\end{cases}$$

precedes $\frac{j}{j+2}$ *in* $\mathcal{F}\bigl(\mathbb{B}(2m),m\bigr)$*; the fraction*

$$\begin{cases}\frac{j\cdot\left(\left\lceil\frac{2m}{j}\right\rceil-1\right)+1}{2}\Big/\frac{(j+2)\left(\left\lceil\frac{2m}{j}\right\rceil-1\right)+1}{2}\ , & \textit{if } \left\lceil\frac{2m}{j}\right\rceil\equiv 0 \pmod 2\ ;\\ \frac{j\cdot\left(\left\lceil\frac{2m}{j}\right\rceil-2\right)+1}{2}\Big/\frac{(j+2)\left(\left\lceil\frac{2m}{j}\right\rceil-2\right)+1}{2}\ , & \textit{if } \left\lceil\frac{2m}{j}\right\rceil\equiv 1 \pmod 2\ ;\end{cases}$$

succeeds $\frac{j}{j+2}$ *in* $\mathcal{F}\bigl(\mathbb{B}(2m),m\bigr)$*.*

Proof. Assertions (i) and (ii) are reformulations of Corollary 3.3(i), made with the help of monotone bijections (3), (4), (7) and (8). Corollary 3.3(ii) leads to assertions (iii) and (iv) by means of bijections (9), (10), (5) and (6). $\square$

5. Three subsequences of adjacent fractions within $\mathcal{F}\bigl(\mathbb{B}(2m),m\bigr)$

Formula (18) implies that the fractions

$$\tfrac{0}{1}<\tfrac{1}{m}<\tfrac{1}{m-1}<\tfrac{1}{m-2}<\cdots<\tfrac{1}{\lceil m/2\rceil} \tag{19}$$

are consecutive in $\mathcal{F}_m$, in the same way as the fractions

$$\tfrac{\lceil m/2\rceil-1}{\lceil m/2\rceil}<\cdots<\tfrac{m-3}{m-2}<\tfrac{m-2}{m-1}<\tfrac{m-1}{m}<\tfrac{1}{1}$$

are consecutive in $\mathcal{F}_m$ thanks to bijection (1); therefore we can clarify the statement of Corollary 4.2 in the following way:

Remark 5.1. (i) *The fractions*

$$\tfrac{0}{1}<\tfrac{1}{m+1}<\tfrac{1}{m}<\tfrac{1}{m-1}<\cdots<\tfrac{1}{\lceil m/2\rceil+1}$$

are consecutive in $\mathcal{F}^{\le\frac{1}{2}}\bigl(\mathbb{B}(2m),m\bigr)$*.*

(ii) *The fractions*

$$\tfrac{\lceil m/2\rceil-1}{2\lceil m/2\rceil-1} < \cdots < \tfrac{m-3}{2m-5} < \tfrac{m-2}{2m-3} < \tfrac{m-1}{2m-1} < \tfrac{1}{2}$$

are consecutive in $\mathcal{F}^{\leq\frac{1}{2}}\bigl(\mathbb{B}(2m),m\bigr)$.

(iii) *The fractions*

$$\tfrac{1}{2} < \tfrac{m}{2m-1} < \tfrac{m-1}{2m-3} < \tfrac{m-2}{2m-5} < \cdots < \tfrac{\lceil m/2\rceil}{2\lceil m/2\rceil-1}$$

are consecutive in $\mathcal{F}^{\geq\frac{1}{2}}\bigl(\mathbb{B}(2m),m\bigr)$.

(iv) *The fractions*

$$\tfrac{\lceil m/2\rceil}{\lceil m/2\rceil+1} < \cdots < \tfrac{m-2}{m-1} < \tfrac{m-1}{m} < \tfrac{m}{m+1} < \tfrac{1}{1}$$

are consecutive in $\mathcal{F}^{\geq\frac{1}{2}}\bigl(\mathbb{B}(2m),m\bigr)$.

Indeed, since the fractions composing sequence (19) *are consecutive in* $\mathcal{F}_m$*, we arrive at conclusions* (i)*,* (ii)*,* (iii) *and* (iv) *with the help of monotone bijections* (4)*,* (8)*,* (6) *and* (10)*, respectively.*

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